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Stability analysis of the Ortloff–Ives equation

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The unstable conclusion drawn by Ortloff & Ives of the cylinder motion in a viscous stream is shown to be incorrect. Furthermore, a region of stable solution is derived in the parameter space defined by the normal and tangential drag coefficients.

1. Introduction

Ortloff & Ives (1969, p. 713) modified the equation derived by Paidousis (1966, p. 737) to model the dynamic motion of a thin flexible cylinder in a viscous stream. This same mathematical model is currently applied to cable-towed arrays (Kennedy 1980). The solution proposed by Ortloff & Ives for their equation predicts a temporally unstable response. If this instability conclusion is true, it implies that the linearized equation derived to describe the low-frequency response of cable-towed array systems is valid for only a limited (i.e. finite) time interval. In this short note the Ortloff & Ives' solution and the resulting temporal instability conclusion are shown to be incorrect. Furthermore, a stable-solution region is derived in the parameter space defined by the normal and tangential drag coefficients of the cable-towed array system.

2. Ortloff & Ives' instability result

The equation of motion of a thin flexible cylinder with zero bending rigidity used by Ortloff & Ives (1969, p. 715) is

$$\frac{\partial^2 y}{\partial t^2}\frac{M+m}{M}+\frac{\partial^2 y}{\partial x^2}\left[u^2-\tfrac{1}{2}C_T\frac{u^2}{D}(L-x)\right]+\frac{\partial y}{\partial x}\frac{u^2}{2D}(C_T+C_N)+2u\frac{\partial^2 y}{\partial t\,\partial x}+\tfrac{1}{2}C_N\frac{u}{D}\frac{\partial y}{\partial t}=0, \quad (2.1)$$

where y = transverse displacement, x = longitudinal distance, t = time, M = virtual mass of fluid per unit length, m = mass of the cylinder per unit length, u = free-stream velocity, L = cylinder length, D = cylinder diameter, C_T = tangential drag coefficient, and C_N = normal drag coefficient. With the following non-dimensionalizations

$$\tau=(t/L)u, \quad \beta=M/(M+m), \quad \xi=x/L, \quad \epsilon=L/D, \quad \eta=y/L, \quad (2.2)$$

and the following notation

$$\left.\begin{aligned} &a=\beta(1-\tfrac{1}{2}C_T\epsilon), \quad b=\tfrac{1}{2}\beta C_T\epsilon),\\ &c=\tfrac{1}{2}(C_T+C_N)\epsilon\beta=b+d,\\ &d=\tfrac{1}{2}C_N\epsilon\beta, \quad \mu=a+b\xi,\end{aligned}\right\} \quad (2.3)$$

equation (2.1) can be transformed into (Ortloff & Ives 1969, p. 716)

$$\frac{\partial^2\eta}{\partial\tau^2}+\mu b^2\frac{\partial^2\eta}{\partial\mu^2}+bc\frac{\partial\eta}{\partial\mu}+2\beta b\frac{\partial^2\eta}{\partial\tau\,\partial\mu}+d\frac{\partial\eta}{\partial\tau}=0. \tag{2.4}$$

Ortloff & Ives (1969, p. 717) claim that the solution of (2.4) (satisfying the boundary conditions specified by them) when ϵ approaches infinity while $\beta\epsilon$ remains bounded, and $C_N/C_T=\frac{1}{2}$, is

$$\eta(\xi,\tau)=\mathscr{R}\left\{\left(\frac{-a}{b}-\xi\right)^{-\frac{1}{4}}(-b)^{-\frac{1}{4}}\sum_{n=1}^{\infty}B_n(\cos\omega_n\tau)\,J_{\frac{1}{2}}\left[\pm 2\omega_n^{\frac{1}{2}}\left(-\frac{a}{b}-\xi\right)^{\frac{1}{2}}b^{-\frac{1}{2}}(\omega_n-id)^{\frac{1}{2}}\right]\right\}, \tag{2.5}$$

where $\mathscr{R}\{.\}=$ real part of $\{.\}$, $J_{\frac{1}{2}}(.)=$ Bessel function of order $\frac{1}{2}$,

$$\omega_n=\frac{id}{2}\pm\frac{1}{2}\left(-\frac{b^2}{a}\,\delta_n^2-d^2\right)^{\frac{1}{2}}, \tag{2.6}$$

$$\delta_n=n\pi\quad(n=0,1,2,\ldots) \tag{2.7}$$

$$=\text{zeros of } J_{\frac{1}{2}}(x),$$

B_n, $n=1,2,\ldots$, are constants to be chosen from the prescribed initial condition $\eta(\xi,0)=\eta_1(\xi)$.

Thus, according to the proof of Ortloff & Ives,

$$\eta(\mu,\tau)=\mathscr{R}\{\cos\omega_1\tau\,V(\mu)\}, \tag{2.8}$$

where

$$V(\mu)=\mu^{-\frac{1}{4}}J_{\frac{1}{2}}[\pm 2i\omega_1^{\frac{1}{2}}\mu^{\frac{1}{2}}(\omega_1-id)^{\frac{1}{2}}] \tag{2.9}$$

is the solution of (2.4) if the initial deflection is chosen as

$$\eta_1(\mu)=V(\mu), \tag{2.10}$$

as ϵ approaches infinity while $\beta\epsilon$ remains bounded. Since ω_1 is always complex-valued (see equation (2.6)), there is an unbounded component in $\cos\omega_1\tau$ (equation (2.8)). This is the basis of the instability result argued by Ortloff & Ives. In the next section, this instability conclusion is shown to be incorrect, while a rather large stable region is derived.

3. Derivation of stable region

The correct solution form of (2.4) is

$$\eta(\mu,\tau)=\mathscr{R}\{e^{i\omega\tau}\,V(\mu)\}, \tag{3.1}$$

where

$$V(\mu)=\mu^{-C_N/2C_T}\,J_{C_N/C_T}[\pm 2i\omega_n^{\frac{1}{2}}\mu^{\frac{1}{2}}(\omega_n-id)^{\frac{1}{2}}], \tag{3.2}$$

as opposed to (2.8) used by Ortloff & Ives. The complex character of ω_n leads Ortloff & Ives to the incorrect conclusion that the motion is unstable for all values of parameters in ω_n.

The boundary condition of $V(-b)=0$ requires that

$$\pm 2i\omega_n^{\frac{1}{2}}\mu^{\frac{1}{2}}(\omega_n-id)^{\frac{1}{2}}=\delta_n, \tag{3.3}$$

where δ_n are zeros of the Bessel function in (3.2). Solving (3.3) for ω_n gives

$$\omega_n=\frac{id}{2}\pm\frac{1}{2}\left(-\frac{b^2}{a}\delta_n^2-d^2\right)^{\frac{1}{2}}. \tag{3.4}$$

Since the zeros of the Bessel function $J_{C_N/C_T}(.)$ are all real (Abramowitz & Stegun 1972), the necessary and sufficient condition for a bounded solution of the form of (3.1) is

$$\mathscr{I}\{\omega_n\} \geqslant 0. \tag{3.5}$$

Therefore, either

$$\sigma_n = -\frac{b^2}{a}\delta_n^2 - d^2 \geqslant 0 \tag{3.6}$$

or

$$\sigma_n < 0 \quad \text{and} \quad d \geqslant (-\sigma_n)^{\frac{1}{2}} \tag{3.7}$$

are the conditions for a stable solution of (2.4). In order to gain more insight from conditions (3.6) and (3.7) the following property is proved.

Property. If $C_T > 2/\epsilon$, then $\eta(\mu, \tau)$ (defined by (3.1)) is bounded.

Proof. If $C_T > 2/\epsilon$ then

$$a = \beta(1 - \tfrac{1}{2}C_T\epsilon) < 0. \tag{3.8}$$

Thus

$$-\frac{b^2}{a}\delta_n^2 \geqslant 0, \tag{3.9}$$

and

$$|\sigma_n|^{\frac{1}{2}} \leqslant d, \tag{3.10}$$

which is condition (3.7). *Q.E.D.*

Note that the physical interpretation of the above property is simply that the dynamic motion of the thin flexible cylinder is stable if the tension along the cylinder is large enough.

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REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1972 *Handbook of Mathematical Functions*. Dover.
KENNEDY, R. M. 1980 *USN J. Underwater Acoustics*.
PAIDOUSIS, M. P. 1966 *J. Fluid Mech.* **26**, 737.
ORTLOFF, C. R. & IVES, J. 1969 *J. Fluid Mech.* **38**, 713.